CONCERNING THE THERMAL EFFECT IN THE FLOW OF AN ELECTRICALLY CONDUCTING FLUID BETWEEN PARALLEL WALLS

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PMM Vol.23, No.5, 1959, pp. 948-950

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(Received 25 November 1958)

This note investigates the flow of a viscous electrically conducting fluid in a plane duct in the presence of a magnetic field and heat transfer. The generation of heat from internal friction, Joule heat, and the dependence of viscosity on temperature are taken into account. An analogous problem in isothermal flow was solved by Hartman [1]; for non-conducting fluids, it was previously considered also in references [2-4].

1. We shall consider steady flow of a non-viscous fluid in the x-direction between infinite parallel walls $z = \pm a$, when a uniform magnetic field H_0 is imposed in a direction normal to it. Just as in the isothermal case the general equations of magnetohydrodynamics permit [1] a solution of the form:

$$H_x = H_x(z), \qquad H_y = 0, \qquad H_z = H_0 = \text{const} \qquad \text{(for the magnetic field)}$$

$$v_x = v(z), \qquad v_y = v_z = 0 \qquad \text{(for the velocities)}$$

$$T = T(z), \qquad p = p(x, z) \qquad \text{(for temperature and pressure)}$$

The functions sought for satisfy the system

$$\frac{d}{dz}\left(\eta \frac{dv}{dz}\right) + \frac{\mu H_0}{4\pi}\frac{dH_x}{dz} = \frac{\partial}{\partial x}\left(p + \frac{\mu H^2}{8\pi}\right) \tag{1.1}$$

$$H_0 \frac{dv}{dz} + \frac{c^2}{4\pi\sigma\mu}\frac{d^2 H_x}{dz^2} = 0 \tag{1.2}$$

$$k\frac{d^2T}{dz^2} + \eta\left(\frac{dv}{dz}\right)^2 + \frac{c^2}{16\pi^2\sigma}\left(\frac{dH_x}{dz}\right)^2 = 0 \tag{1.3}$$

$$\frac{\partial}{\partial y}\left(p + \frac{\mu H^2}{8\pi}\right) = 0, \qquad \frac{\partial}{\partial z}\left(p + \frac{\mu H^2}{8\pi}\right) = 0 \tag{1.4}$$

Here c is the velocity of light, $\eta = \eta(T)$, k = const, σ = const, μ = const correspond to the viscosity, the thermal conductivity, the electrical conductivity and the magnetic permeability of the fluid respectively.

We limit ourselves to the consideration of problems with the simplest boundary conditions of the form

$$v(\pm a) = H_x(\pm a) = T(\pm a) = 0$$

From equation (1.4) it follows that $p + \mu H^2/8\pi = f(x)$, and from (1.1), since the left side of the equation depends only on z, that

$$\frac{d}{dx}\left(p + \frac{\mu H^2}{8\pi}\right) = \text{const} = p^*$$

We introduce the dimensionless variables

$$\xi = \frac{z}{a}, \quad u = \frac{\eta_0}{p^* a^2} v, \quad h = \frac{H_x}{H_0}, \quad \theta = \frac{k\eta_0}{p^{*2} a^4} T, \quad \psi = \frac{\eta_0}{\eta} \tag{1.5}$$

where $\eta_0 = \eta_{T=0}$, and the parameters

$$A = \mu H_0^2 / 4\pi a p^*, \qquad B = c^2 \eta_0 / 4\pi\sigma\mu a^3 p^*$$

In the new system of variables, (1.1)-(1.3) can be written in the form

$$\left(\frac{u'}{\psi}\right)' + Ah' = 1, \qquad u' + Bh'' = 0, \qquad \theta'' + \frac{1}{\psi} u'^2 + ABh'^2 = 0 \tag{1.6}$$

in which $u(\pm 1) = h(\pm 1) = \theta(\pm 1) = 0, \qquad \psi(\pm 1) = 1.$

The first two equations of the system are integrated directly:

$$\frac{u'}{\psi} + Ah = \xi + C_1, \qquad u + Bh' = C_2 \tag{1.7}$$

From the condition of symmetry, $C_1 = 0$; multiplying the first of the equations (1.7) by u' and the second by Ah', we obtain

$$\frac{u'^2}{\psi} = \xi u' - Ahu'; \qquad ABh'^2 = (C_2 - u) Ah'$$

Introducing the quantity $\tau = \xi - Ah$, with the aid of the second equation (1.6) we transform these equations to the form

$$\frac{u'^2}{\psi} = \frac{B}{A} \tau\tau'', \qquad ABh'^2 = \frac{B}{A}(1 - \tau')^2 \tag{1.8}$$

Consequently, in plase of the third of equations (1.6) we have

$$\theta'' + \frac{B}{A}(\tau\tau'' + 1 - 2\tau' + \tau'^2) = 0 \tag{1.9}$$

Integrating and again, in the light of symmetry, putting the constant equal to zero, we arrive at the relation between $\theta(\xi)$ and $\tau(\xi)$:

$$\theta' + \frac{B}{A}(\xi - 2\tau + \tau\tau') = 0 \tag{1.10}$$

A second equation is obtained by eliminating u from equation (1.7)

$$\psi(\theta) = \frac{B\tau''}{A\tau} \tag{1.11}$$

Boundary conditions for the variable $\tau(\xi)$ are $\tau(+1) = +1$.

Since the function $\psi(\theta)$ is ordinarily expressed as a power series in θ, convergent within a sufficiently wide interval, the solution of system (1.10) and (1.11) may be represented by a power series.

2. The solution of equations (1.10) and (1.11) in closed form is obtained only in the case of constant viscosity, when $\psi(\theta) \equiv 1$. From (1.11) and the boundary conditions for τ we obtain (see [1])

$$\tau = \frac{\operatorname{sh} \lambda\xi}{\operatorname{sh} \lambda}, \qquad \lambda = \sqrt{\frac{A}{B}} = \frac{\mu H_0 a}{c} \sqrt{\frac{\sigma}{\eta_0}} \tag{2.1}$$

Therefore equation (1.10) takes the form

$$\theta' + \frac{1}{\lambda^2}\left(\xi - 2\frac{\operatorname{sh} \lambda\xi}{\operatorname{sh} \lambda} + \frac{\lambda}{2 \operatorname{sh}^2 \lambda} \operatorname{sh}^2 \lambda\xi\right) = 0 \tag{2.2}$$

and is easily integrated; its integral satisfying the boundary conditions, is

$$\theta = \frac{1}{2\lambda^2}(1 - \xi^2) - \frac{2}{\lambda^3 \operatorname{sh} \lambda}(\operatorname{ch} \lambda - \operatorname{ch} \lambda\xi) + \frac{1}{4\lambda^2 \operatorname{sh}^2 \lambda}(\operatorname{ch} 2\lambda - \operatorname{ch} 2\lambda\xi) \tag{2.3}$$

The maximum value of temperature is reached for $\xi = 0$:

$$\theta_m = \theta(0) = \frac{1}{\lambda^2}\left(1 - \frac{\operatorname{th} \lambda/2}{\lambda/2}\right) \tag{2.4}$$

Proceeding from the corresponding incomplete energy, equations , it is easy to construct also solutions for θ, taking into account only one of the thermal effects: the Joule heat $\theta^{(1)}$ or the frictional heat $\theta^{(2)}$.

$$\theta^{(1)} = \left(\frac{1}{2\lambda^2} + \frac{1}{4 \operatorname{sh}^2 \lambda}\right)(1 - \xi^2) - \frac{2}{\lambda^3 \operatorname{sh} \lambda}(\operatorname{ch} \lambda - \operatorname{ch} \lambda\xi) + \frac{1}{8\lambda^2 \operatorname{sh}^2 \lambda}(\operatorname{ch} 2\lambda - \operatorname{ch} 2\lambda\xi) \tag{2.5}$$

$$\theta^{(2)} = -\frac{1}{4 \operatorname{sh}^2 \lambda}(1 - \xi^2) + \frac{1}{8\lambda^2 \operatorname{sh}^2 \lambda}(\operatorname{ch} 2\lambda - \operatorname{ch} 2\lambda\xi) \tag{2.6}$$

The dependence of the maximum temperatures $\theta^{(1)}$ and $\theta^{(2)}$ on the parameter λ has the form

$$\theta_m^{(1)} = \frac{1}{\lambda^2}\left(\frac{3}{4} - \frac{\operatorname{th} \lambda/2}{\lambda/2}\right) + \frac{1}{4 \operatorname{sh}^2 \lambda}, \qquad \theta_m^{(2)} = \frac{1}{4}\left(\frac{1}{\lambda^2} - \frac{1}{\operatorname{sh}^2 \lambda}\right) \tag{2.7}$$

From this it is possible to obtain the following boundary relations:

$$\lim \theta_m^{(1)} = 0, \quad \lim \theta_m^{(2)} = \frac{1}{12} \quad \text{as } \lambda \to 0, \qquad \lim \theta_m^{(1)} = \lim \theta_m^{(2)} = 0 \quad \text{as } \lambda \to \infty$$

Comparison of formulas (2.7) shows that for small λ the principal role in the increase of temperature is played by internal friction, but for sufficiently large λ the situation is reversed.

3. We now consider the case where the viscosity is related to the temperature by a hyperbolic dependence

$$\eta = \eta_0 \frac{1}{1 + \alpha^2 T}, \quad \text{или} \quad \psi(\theta) = 1 + \frac{p^{*2} a^4 \alpha^2}{k\eta_0}\theta = 1 + \beta^2\theta \tag{3.1}$$

Equations (1.10) and (1.11) are then written in the form

$$\tau'' - \lambda^2\psi\tau = 0, \qquad \psi' + \frac{\beta^2}{\lambda^2}(\xi - 2\tau + \tau\tau') = 0 \tag{3.2}$$

Considerations of symmetry and the form of these equations lead to the conclusion that $\tau(\xi)$ is an odd function, and that $\psi(\xi)$ is an even one. Therefore, we shall express them as series:

$$\tau = \sum_{i=0}^{\infty} a_{2i+1}\xi^{2i+1}; \qquad \psi = \sum_{i=0}^{\infty} b_{2i}\xi^{2i} \tag{3.3}$$

Substituting series (3.3) into equations (3.2), we obtain the recurrence formulas

$$(2i+3)(2i+2)a_{2i+3} - \lambda^2 \sum_{m=0}^{m=i} a_{2m+1} b_{2(i-m)} = 0, \quad (i \geqslant 0) \tag{3.4}$$

$$(2i+2)b_{2i+2} - \frac{\beta^2}{\lambda^2}\Big(2a_{2i+1} - (i+1)\sum_{m=0}^{m=i} a_{2m+1}a_{2(i-m)+1}\Big) = 0, \ (i>0) \tag{3.5}$$

which give us means for the computation of the coefficients of the expansions (3.3) as functions of a_1, b_0 and b_2. For this purpose, a_1 and b_0 are determined from the boundary conditions for τ and ψ, which here have the form:

$$\sum_{i=0}^{\infty} a_{2i+1} = 1, \qquad \sum_{i=0}^{\infty} b_{2i} = 1, \qquad b_2 = \frac{(1-a_1)^2\beta^2}{\lambda^2} \tag{3.6}$$

Combining equations (3.4) and (3.5) and the formula for b_2, it is possible to obtain a relation containing only a_{2s+1}:

$$a_{2i+3} = \frac{\beta^2}{(2i+3)(2i+2)}\left[\frac{\lambda^2 b_0}{\beta^2}a_{2i+1} - a_{2i-1} - \sum_{m=0}^{m=i-1} a_{2m+1} \times \right. \tag{3.7}$$

$$\left. \times \left(\frac{a_{2(i-m)-1}}{i-m} - \frac{1}{2}\sum_{n=0}^{n=i-m-1} a_{2n+1}a_{2(i-m-n)-1}\right)\right] \qquad (i>0)$$

The question of convergence of the series (3.3) can hardly be studied fully. However, for sufficiently small λ and β it is possible to demonstrate that the series (3.3) are uniformly convergent in the interval $(-\rho, +\rho)$, where ρ is some number greater than unity. In the same context, if for $2 \leqslant i \leqslant n$ the inequality $|a_{2i+1}| < M\epsilon^i$ is valid, where $0 < \epsilon < 1$ and $M > b_0$, $M > a_1$, then from formula (3.7), $|a_{2n+3}| < M\epsilon^{n+1}$ can be easily obtained for sufficiently small λ and β. If, in addition to the smallness of λ and β the inequality $|a_5| < M\epsilon^2$ is also valid, then by induction, $|a_{2i+1}| < M\epsilon^i$ for all $i > 2$. Then the radius of convergence of the first of the series is

$$\rho = \lim_{n\to\infty}(a_{2n+1})^{-1/n} \geqslant \frac{1}{\varepsilon} > 1 \tag{3.8}$$

The convergence of the second series is investigated with the aid of the same evaluation, proceeding from (3.5).

Similarly it is possible to construct solutions of our problem in the form of a uniformly convergent series, taking account of only one of the thermal effects just as in Section 2.

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Translated by M.V.M.